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Symmetrical Rigid Body Parameterizations For Humanoid Robots

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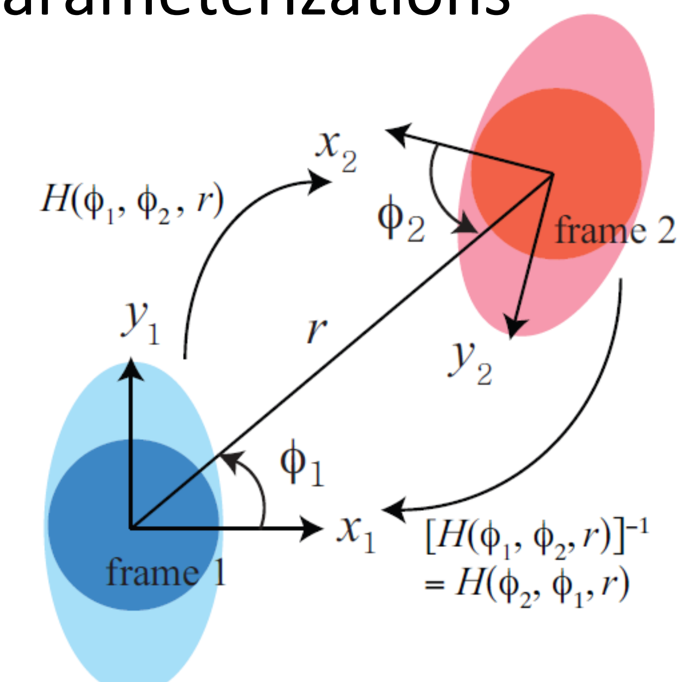
Content

- Background
- Reviews on Symmetrical Parameterizations
- Possible Forms of Symmetrical Parameterizations
- Product Formulas for Symmetrical Parameterizations
- Comparisons with Traditional Parameterizations
- Conclusions

Rigid-body Representations

- Describe orientations and positions of robotic systems
  - Standard approach: relative to the fixed world frame
  - Multi-robot system: relative motions between individual robots in the group
- More beneficial: "Symmetrical" Parameterizations
  - Parameters for relative motions and the inverse between a pair of robots in the group are calculated in the same way
  - Enhance the evaluation of relative rigid-body motions



Rotations: Euler-Angles Parameterizations

- Definitions and Notations:
  - 3D Rotation: $RR^T = R^T R = I_{3\times 3}$ and $detR = +1$
  - Rotations around x, y, z axes: $R_{x,y,z}(\theta)$
- Commonly used Euler-Angles:
  - ZXZ: $R_{ZXZ}(\alpha,\beta,\gamma)$=$R_z(\alpha)R_x(\beta)R_z(\gamma)$
  - ZYZ: $R_{ZYZ}(\alpha,\beta,\gamma)$=$R_z(\alpha)R_y(\beta)R_z(\gamma)$
- Inversions:
  - ZXZ: $(R_{ZXZ}(\alpha,\beta,\gamma))^T$=$R_z(-\gamma)R_x(-\beta)R_z(-\alpha)$
  - ZYZ: $(R_{ZYZ}(\alpha,\beta,\gamma))^T$=$R_z(-\gamma)R_y(-\beta)R_z(-\alpha)$
- <u>Not</u> Symmetrical:
  - range of β cannot be matched back correctly in inverse operations ($0 \le \beta \le \pi$).

Transference Rotation Matrix

- Notations:
  - $R(\boldsymbol{a},\boldsymbol{b}) \doteq \exp\left(\theta_{ab}\cdot\frac{\widehat{\boldsymbol{a}\times\boldsymbol{b}}}{||\boldsymbol{a}\times\boldsymbol{b}||}\right)$

$$= I_{3\times 3} + \widehat{\boldsymbol{a}\times\boldsymbol{b}} + \frac{(1-\boldsymbol{a}\cdot\boldsymbol{b})}{||\boldsymbol{a}\times\boldsymbol{b}||^2}(\widehat{\boldsymbol{a}\times\boldsymbol{b}})^2$$

  where, $\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{S}^2$ and $\sin\theta_{ab} = ||\boldsymbol{a}\times\boldsymbol{b}||$, $\cos\theta_{ab} = \boldsymbol{a}\cdot\boldsymbol{b}$

$\widehat{\boldsymbol{a}} = -\widehat{\boldsymbol{a}} \in \boldsymbol{R}^{3\times 3}$ is a skew-symmetric matrix

- Properties:
  - $R(\boldsymbol{a},\boldsymbol{b})\boldsymbol{a} = \boldsymbol{b}$
  - $(R(\boldsymbol{a},\boldsymbol{b}))^{-1} = R(\boldsymbol{b},\boldsymbol{a}) = R(-\boldsymbol{b},-\boldsymbol{a})$
  - $R(\boldsymbol{a},-\boldsymbol{b}) = R(-\boldsymbol{a},\boldsymbol{b})$
  - $AR(\boldsymbol{a},\boldsymbol{b})A^T = R(A\boldsymbol{a},A\boldsymbol{b})$, $A\in SO(3)$
  - $R(\boldsymbol{a},\boldsymbol{b})\mathrm{rot}(\boldsymbol{a},\theta) = \mathrm{rot}(\boldsymbol{b},\theta)R(\boldsymbol{a},\boldsymbol{b})$, where $\mathrm{rot}(\boldsymbol{a},\theta) = e^{\theta\widehat{\boldsymbol{a}}}$ denotes a rotation along axis $\boldsymbol{a}$ with angle θ

Symmetrical Parameterizations: SO(3)

- Explicit Forms:
  - $R_{12}(\boldsymbol{u}_1, \boldsymbol{u}_2, \nu) \doteq \mathrm{rot}(\boldsymbol{u}_1, \frac{\nu}{2}) R(\boldsymbol{u}_1, \boldsymbol{u}_2) \mathrm{rot}(\boldsymbol{u}_2, \frac{-\nu}{2})$, or
  - $R_{12}(\boldsymbol{u}_1, \boldsymbol{u}_2, \alpha, \beta) \doteq \mathrm{rot}(\boldsymbol{u}_1, -\alpha) R(\boldsymbol{u}_2, \boldsymbol{u}_1) \mathrm{rot}(\boldsymbol{u}_2, \beta)$
- Inverse Operations:
  - $R_{12}^{-1} = \mathrm{rot}\left(\boldsymbol{u}_2, \frac{\nu}{2}\right) R(\boldsymbol{u}_2, \boldsymbol{u}_1) \mathrm{rot}\left(\boldsymbol{u}_1, \frac{-\nu}{2}\right) = R_{21}$, or
  - $R_{12}^{-1} = \mathrm{rot}(\boldsymbol{u}_2, -\beta) R(\boldsymbol{u}_1, \boldsymbol{u}_2) \mathrm{rot}(\boldsymbol{u}_1, \alpha) = R_{21}$
- *Note:* $\boldsymbol{u}_1$ *can be fixed as a unit vector* $[1,0,0]^T$ *to ensure 3 DOFs in SO(3).*

$(\boldsymbol{u}_1, \boldsymbol{u}_2, \nu) \leftrightarrow (\boldsymbol{u}_2, \boldsymbol{u}_1, \nu)$

$(\boldsymbol{u}_1, \boldsymbol{u}_2, \alpha, \beta) \leftrightarrow (\boldsymbol{u}_2, \boldsymbol{u}_1, \beta, \alpha)$

Possible Forms

- By switching the role of parameters, one can get possible forms other than the ones on the definition:

$$
\begin{array}{ll}
\mathrm{rot}(\mathbf{u}_1, -\nu/2) R(\mathbf{u}_1, \mathbf{u}_2) \mathrm{rot}(\mathbf{u}_2, \nu/2) & \mathrm{rot}(\mathbf{u}_1, -\nu/2) R(\mathbf{u}_2, \mathbf{u}_1) \mathrm{rot}(\mathbf{u}_2, \nu/2) \\
\mathrm{rot}(\mathbf{u}_1, \nu/2) R(\mathbf{u}_1, -\mathbf{u}_2) \mathrm{rot}(\mathbf{u}_2, -\nu/2) & \mathrm{rot}(\mathbf{u}_1, \nu/2) R(\mathbf{u}_2, -\mathbf{u}_1) \mathrm{rot}(\mathbf{u}_2, -\nu/2) \\
\mathrm{rot}(\mathbf{u}_1, -\nu/2) R(\mathbf{u}_1, -\mathbf{u}_2) \mathrm{rot}(\mathbf{u}_2, \nu/2) & \mathrm{rot}(\mathbf{u}_1, -\nu/2) R(\mathbf{u}_2, -\mathbf{u}_1) \mathrm{rot}(\mathbf{u}_2, \nu/2) \\
\mathrm{rot}(\mathbf{u}_1, \nu/2) R(-\mathbf{u}_1, \mathbf{u}_2) \mathrm{rot}(\mathbf{u}_2, -\nu/2) & \mathrm{rot}(\mathbf{u}_1, \nu/2) R(-\mathbf{u}_2, \mathbf{u}_1) \mathrm{rot}(\mathbf{u}_2, -\nu/2) \\
\mathrm{rot}(\mathbf{u}_1, -\nu/2) R(-\mathbf{u}_1, \mathbf{u}_2) \mathrm{rot}(\mathbf{u}_2, \nu/2) & \mathrm{rot}(\mathbf{u}_1, -\nu/2) R(-\mathbf{u}_2, \mathbf{u}_1) \mathrm{rot}(\mathbf{u}_2, \nu/2).
\end{array}
$$

$$
\begin{array}{ll}
\mathrm{rot}(\mathbf{u}_2, -\nu/2) R(\mathbf{u}_1, \mathbf{u}_2) \mathrm{rot}(\mathbf{u}_1, \nu/2) & \mathrm{rot}(\mathbf{u}_2, -\nu/2) R(\mathbf{u}_2, \mathbf{u}_1) \mathrm{rot}(\mathbf{u}_1, \nu/2) \\
\mathrm{rot}(\mathbf{u}_2, \nu/2) R(\mathbf{u}_1, -\mathbf{u}_2) \mathrm{rot}(\mathbf{u}_1, -\nu/2) & \mathrm{rot}(\mathbf{u}_2, \nu/2) R(\mathbf{u}_2, -\mathbf{u}_1) \mathrm{rot}(\mathbf{u}_1, -\nu/2) \\
\mathrm{rot}(\mathbf{u}_2, -\nu/2) R(\mathbf{u}_1, -\mathbf{u}_2) \mathrm{rot}(\mathbf{u}_1, \nu/2) & \mathrm{rot}(\mathbf{u}_2, -\nu/2) R(\mathbf{u}_2, -\mathbf{u}_1) \mathrm{rot}(\mathbf{u}_1, \nu/2) \\
\mathrm{rot}(\mathbf{u}_2, \nu/2) R(-\mathbf{u}_1, \mathbf{u}_2) \mathrm{rot}(\mathbf{u}_1, -\nu/2) & \mathrm{rot}(\mathbf{u}_2, \nu/2) R(-\mathbf{u}_2, \mathbf{u}_1) \mathrm{rot}(\mathbf{u}_1, -\nu/2) \\
\mathrm{rot}(\mathbf{u}_2, -\nu/2) R(-\mathbf{u}_1, \mathbf{u}_2) \mathrm{rot}(\mathbf{u}_1, \nu/2) & \mathrm{rot}(\mathbf{u}_2, -\nu/2) R(-\mathbf{u}_2, \mathbf{u}_1) \mathrm{rot}(\mathbf{u}_1, \nu/2).
\end{array}
$$

Product Formula: SO(3)

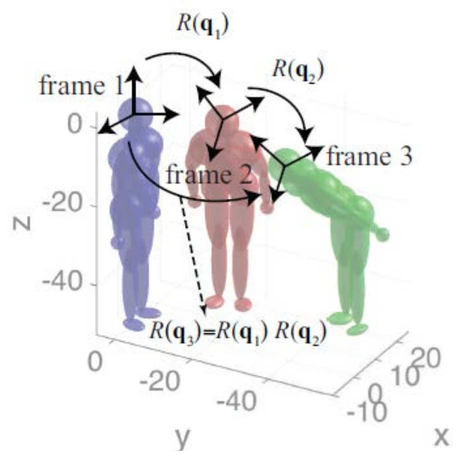
- Problem to be Solved:
  - $R(\boldsymbol{u}, \boldsymbol{v}, \vartheta, \varphi) = R(\boldsymbol{a}, \boldsymbol{b}, \alpha, \beta) R(\boldsymbol{b}, \boldsymbol{c}, \beta, \gamma)$
- Product Formulas in Explicit Form:
  - $[RHS] = \text{rot}(\boldsymbol{a}, -\alpha) \underbrace{R(\boldsymbol{b}, \boldsymbol{a}) R(\boldsymbol{c}, \boldsymbol{b})} \text{rot}(\boldsymbol{c}, \gamma)$

$$\text{rot}(\boldsymbol{a}, \theta_1) \underbrace{R(\boldsymbol{c}, \boldsymbol{a}) \text{rot}(\boldsymbol{c}, \theta_2)}$$

$$\text{rot}(\boldsymbol{a}, \theta_2)\, R(\boldsymbol{c}, \boldsymbol{a})$$

$$= \text{rot}\big(\boldsymbol{a}, -(\alpha - \theta)\big) R(\boldsymbol{c}, \boldsymbol{a}) \text{rot}(\boldsymbol{c}, \gamma)$$

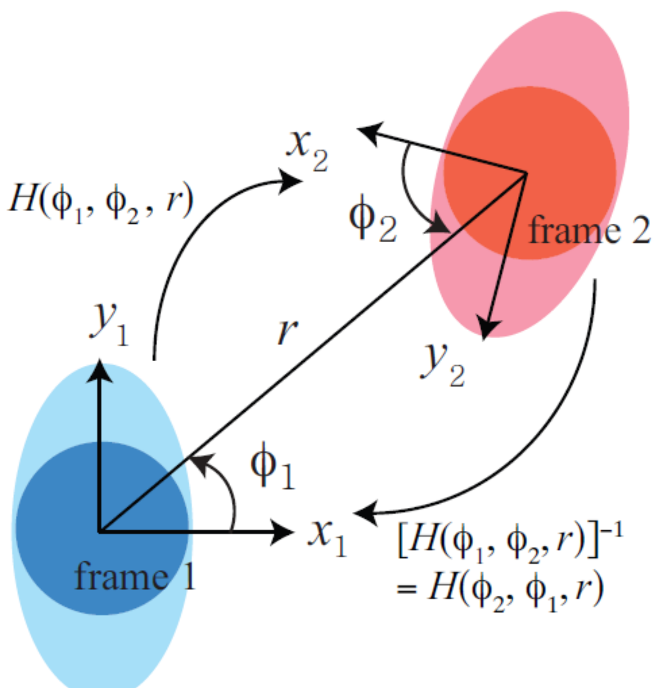
$$= R(\boldsymbol{a}, \boldsymbol{c}, (\alpha - \theta), \gamma)$$



Note: $\theta = \theta_1 + \theta_2 = ||\log^{\vee}(R(\boldsymbol{b}, \boldsymbol{a}) R(\boldsymbol{c}, \boldsymbol{b}) R^T(\boldsymbol{c}, \boldsymbol{a}))||$

Symmetrical Parameterizations: SE(2)

- Definition: Homogeneous Matrix
  - $H(\phi_1, \phi_2, r) =$
    $$\begin{pmatrix} -\cos(\phi_1 - \phi_2) & \sin(\phi_1 - \phi_2) & r\cos(\phi_1) \\ -\sin(\phi_1 - \phi_2) & -cos(\phi_1 - \phi_2) & r\sin(\phi_1) \\ 0 & 0 & 1 \end{pmatrix}$$



Symmetrical Parameterizations: SE(2)

- Inversions:
  - $H^{-1}(\phi_1, \phi_2, r) = H(\phi_2, \phi_1, r)$

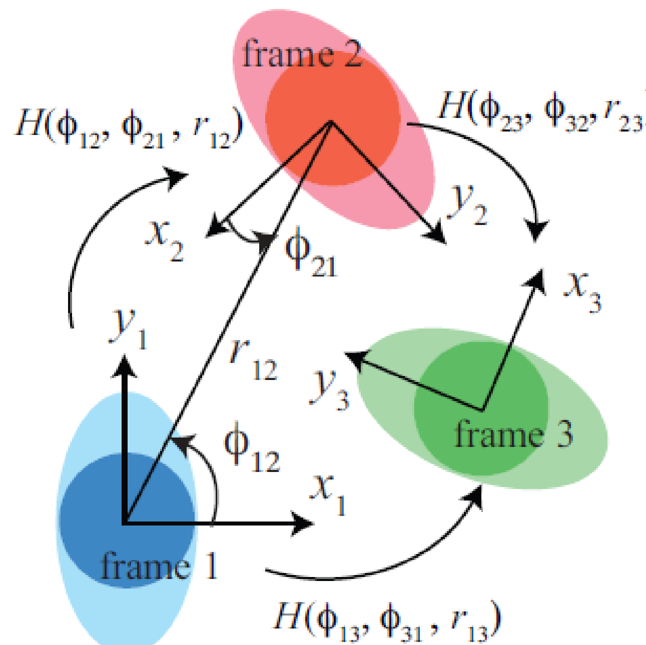
$$= \begin{pmatrix} -cos(\phi_2 - \phi_1) & sin\,(\phi_2 - \phi_1) & rcos(\phi_2) \\ -sin(\phi_2 - \phi_1) & -cos(\phi_2 - \phi_1) & rsin(\phi_2) \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -\cos(\phi_1 - \phi_2) & \sin(\phi_1 - \phi_2) \\ -\sin(\phi_1 - \phi_2) & -\cos(\phi_1 - \phi_2) \end{pmatrix}^T$$

$$r\cos(\phi_1)(-\cos(\phi_1 - \phi_2)) + r\sin(\phi_2)\sin\,(\phi_2 - \phi_1)$$
$$r\cos(\phi_1)(-\sin(\phi_1 - \phi_2)) + r\sin(\phi_2)(-\cos(\phi_1 - \phi_2))$$

Product Formula: SE(2)

- Solving for $(\phi_{13}, \phi_{31}, r_{13})$ that satisfies:
  - $H(\phi_{13}, \phi_{31}, r_{13}) = H(\phi_{12}, \phi_{21}, r_{12})H(\phi_{23}, \phi_{32}, r_{23})$
- Product Formula:
  - $r_{13} = \sqrt{r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}\cos(\phi_{21} - \phi_{23})}$
  - $\phi_{13} = \text{Atan2}(x_{13}, y_{13})$
    - $x_{13} = r_{12}\cos\phi_{12} - r_{23}\cos(\phi_{12} + \phi_{23} - \phi_{21})$
    - $x_{13} = r_{12}\sin\phi_{12} - r_{23}\sin(\phi_{12} + \phi_{23} - \phi_{21})$
  - $\phi_{31} = \pi + \phi_{13} - (\phi_{12} - \phi_{21} + \phi_{23} - \phi_{32})$



Symmetrical Parameterizations: SE(3)

- Definition: Homogeneous Matrix
  - $H(\boldsymbol{u}_1, \boldsymbol{u}_2, \nu, r_{12}) = \begin{pmatrix} R(\boldsymbol{u}(\theta_1, \phi_1), \boldsymbol{u}(\theta_2, \phi_2), \nu) & r_{12}\boldsymbol{u}(\theta_1, \phi_1) \\ \boldsymbol{0}^T & 1 \end{pmatrix}$
    - Note: $\boldsymbol{u}_1$ is no longer a fixed vector.
  - Specifically choose one of the parameterizations:
    - $R(\boldsymbol{u}_1, \boldsymbol{u}_2, \nu) \doteq \mathrm{rot}(\boldsymbol{u}_1, \pm\frac{\nu}{2}) R(\boldsymbol{u}_2, -\boldsymbol{u}_1) \mathrm{rot}(\boldsymbol{u}_2, \mp\frac{\nu}{2})$
- Matrix Inverse:
  - Rotation part: $R^T(\boldsymbol{u}_1, \boldsymbol{u}_2, \nu) = R(\boldsymbol{u}_2, \boldsymbol{u}_1, \nu)$
  - Translation part:

$$-R^T(\boldsymbol{u}_1, \boldsymbol{u}_2, \nu) r_{12} \boldsymbol{u}_1$$
$$= -r_{12} \mathrm{rot}\left(\boldsymbol{u}_2, \pm\frac{\nu}{2}\right) R(\boldsymbol{u}_1, -\boldsymbol{u}_2) \mathrm{rot}\left(\boldsymbol{u}_1, \mp\frac{\nu}{2}\right) \boldsymbol{u}_1$$
$$= -r_{12} \mathrm{rot}\left(\boldsymbol{u}_2, \pm\frac{\nu}{2}\right) R(\boldsymbol{u}_1, -\boldsymbol{u}_2) \boldsymbol{u}_1 = -r_{12} \mathrm{rot}\left(\boldsymbol{u}_2, \pm\frac{\nu}{2}\right)(-\boldsymbol{u}_2) = r_{21} \boldsymbol{u}_2$$

Comparisons: Product Formulas on SO(3)

- Traditional Euler Angle representations
  - Extract angles from the explicit product matrix
  - Different conventions for different Euler Angle representations
- Symmetrical Parameterizations
  - resulting directly from input parameters

Method	Symmetrical	Traditional (ZYZ Euler Angles)
Parameters	$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \alpha, \beta, \gamma$	$\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$
Product Formula Representations	$R(\boldsymbol{u}, \boldsymbol{v}, \vartheta, \varphi)$	$R(\alpha, \beta, \gamma)$
Results	$\boldsymbol{u} = \boldsymbol{a}$, $\boldsymbol{v} = \boldsymbol{c}$, $\vartheta = \alpha - \theta$, $\varphi = \gamma$	$\alpha = \mathrm{Atan2}(r_{13}, r_{23})$, $\beta = \cos^{-1}(r_{33})$, $\gamma = \mathrm{Atan2}(-r_{31}, r_{32})$

Note:
$\theta = ||\log^{\vee}(R(\boldsymbol{b}, \boldsymbol{a})R(\boldsymbol{c}, \boldsymbol{b})R^T(\boldsymbol{c}, \boldsymbol{a}))||$, r_{ij} is the i,j-th element in the resulting rotation matrix

Comparisons: Inverse Operations on SE(2)

- Traditional Approach:
  - Known:
    - $H(\theta_{13}, \phi_{13}, r_{13}) = \begin{pmatrix} R(\phi_{13}) & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix}$
  - Inversion:
    - $H(\theta_{31}, \phi_{31}, r_{31}) = \begin{pmatrix} R^T(\phi_{13}) & -R^T(\phi_{13})\boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix}$

Extract the angle ϕ_{31}

Extract the angle θ_{31}

- Symmetrical Parameterization:

$\boldsymbol{H}(\phi_{13}, \phi_{31}, r_{13})$ ⟷ Inverse ⟷ $H(\phi_{31}, \phi_{13}, r_{31})$

Conclusions

- The concept of Symmetrical Parameterizations is reviewed;
- Possible forms of Symmetrical Parameterizations are studied in SO(3) case;
- Product formulas in both rotation and rigid body motion cases are derived;
- Symmetrical Parameterizations are simpler and more convenient than conventional representations in describing relative motions between robots.

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