ASME International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
42nd Mechanisms and Robotics Conference (MR)

Quebec City, Quebec, Canada
August 29, 2018

Lower Bounds of the Allowable Motions of One N-dimensional Ellipsoid Contained in Another

Sipu Ruan [1], Jianzhong Ding [2], Gregory S. Chirikjian [1]

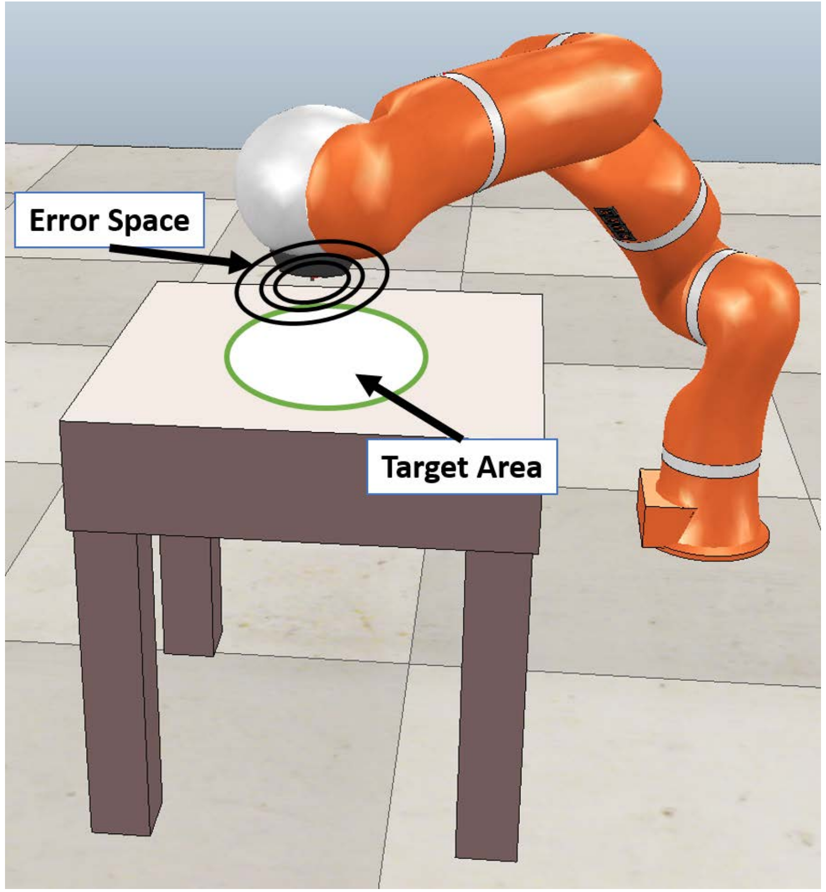
[1] Dept. of Mechanical Engineering, The Johns Hopkins University
[2] Dept. of Mechanical Engineering, Beihang University

Content

- Motivations and mathematical preliminary
- Algebraic Condition of Containment and Convex Lower Bound
- Minkowski Difference and Geometric Lower Bound
- Examples and comparisons for the proposed lower bounds in 2D/3D
- Potential applications

Motivations

- A “pick-and-place” task made by a robot manipulator with uncertainties at the end effector
  - Check if the whole error space is fully contained in the target: *containment checking*
  - Compute how much error the end effector can tolerate in a successful assembly trial: *volume of motions in Configuration space*



Mathematical Preliminary

- Expressions of ellipsoid:
  - Implicit: $(\mathbf{x}-\mathbf{t})^T A(\mathbf{x}-\mathbf{t}) = 1$, $A = R\Lambda^{-2}(\mathbf{a})R^T$
    - $R \in \mathrm{SO}(n), \mathbf{t} \in \mathbb{R}^n$
  - Explicit: $\mathbf{x} = R\Lambda(\mathbf{a})\mathbf{u} + \mathbf{t}$
  - A: the smaller moving ellipsoid; B: the larger fixed ellipsoid
- Configurations:
  - Pose Change Group (PCG): $(R, \mathbf{t}) \in \mathrm{PCG}(n) \doteq \mathrm{SO}(n) \times \mathbb{R}^n$
  - Configurations as elements in the Lie Algebra: $\boldsymbol{\xi} = [\boldsymbol{\omega}^T, \mathbf{t}^T]^T \in \mathbb{R}^{n(n+1)/2}$
    - $\boldsymbol{\omega} = \log^\vee(R) \in \mathbb{R}^{n(n-1)/2}$

Convex Lower Bound based on the First-order Algebraic Condition of Containment

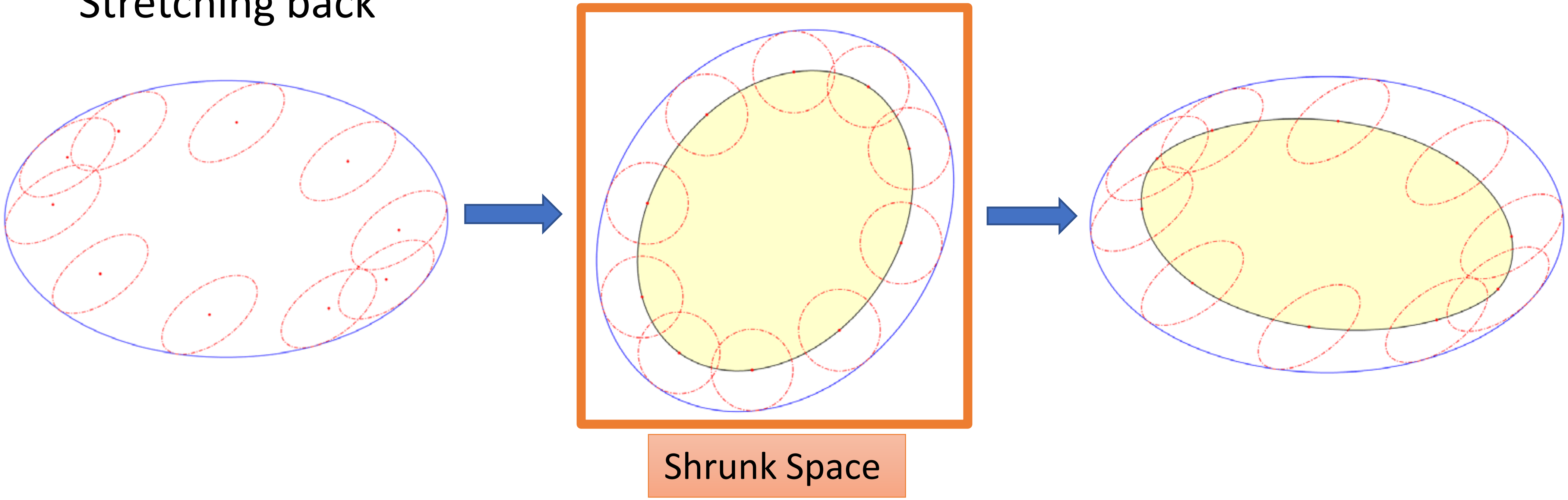
- Algebraic Condition of Containment:
  - Exact: $(R_a \Lambda(\mathbf{a})\mathbf{u} + \boldsymbol{t}_a)^T \Lambda^{-2}(\boldsymbol{b})(R_a \Lambda(\mathbf{a})\mathbf{u} + \boldsymbol{t}_a) \leq 1$
  - First-order Approximation:
    - $C_{\mathbf{u}}(\boldsymbol{\xi}) = \boldsymbol{\xi}^T H(\mathbf{u})\boldsymbol{\xi} + \mathbf{h}^T(\mathbf{u})\boldsymbol{\xi} + c(\mathbf{u})$ ⟺ $\max_{\forall \mathbf{u}_i} C_i(\boldsymbol{\xi}) \leq 1$
      - Small angle assumption: $R_a = \exp(\widehat{\omega}_a) \approx \mathbb{I} + \widehat{\omega}_a$

Convex Lower Bound based on the First-order Algebraic Condition of Containment

- Convexity of the First-order Algebraic Condition of Containment
  - $C_i(\alpha\boldsymbol{\xi}_1 + (1-\alpha)\boldsymbol{\xi}_2) - [\alpha C_i(\boldsymbol{\xi}_1) + (1-\alpha)C_i(\boldsymbol{\xi}_2)]$
    $= -\alpha(1-\alpha)[(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T H(\mathbf{u}_i)(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)] < 0, \forall\alpha \in [0,1]$
    - $H(\mathbf{u}_i)$ is symmetric positive-definite
  - Maximization preserves convexity
- Polyhedron C-space as the Convex Lower Bound
  - Valid configurations stays in the convex hull of some extreme configurations
  - Vertex Selections:
    - Maximum distance on each axis of the C-space
    - Maximum distance to the origin, maximum magnitude
    - Random configurations

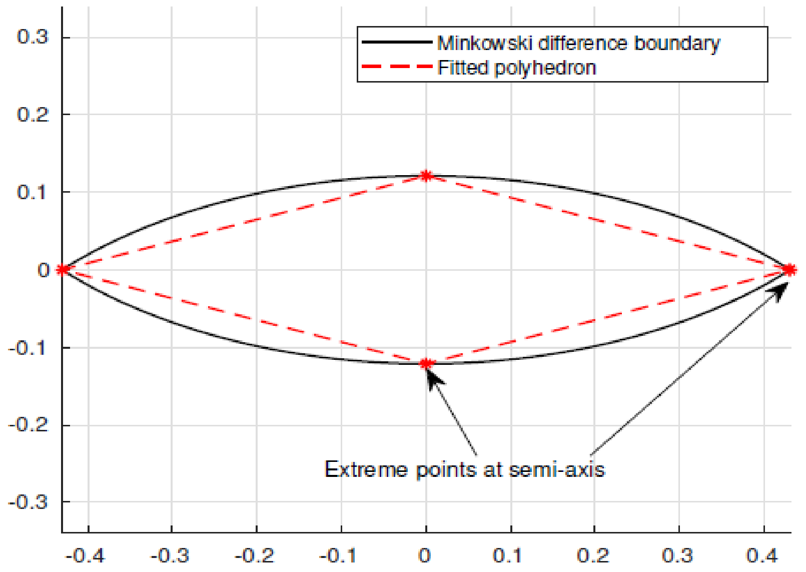
Geometric Lower Bound based on the Closed-form Minkowski Difference

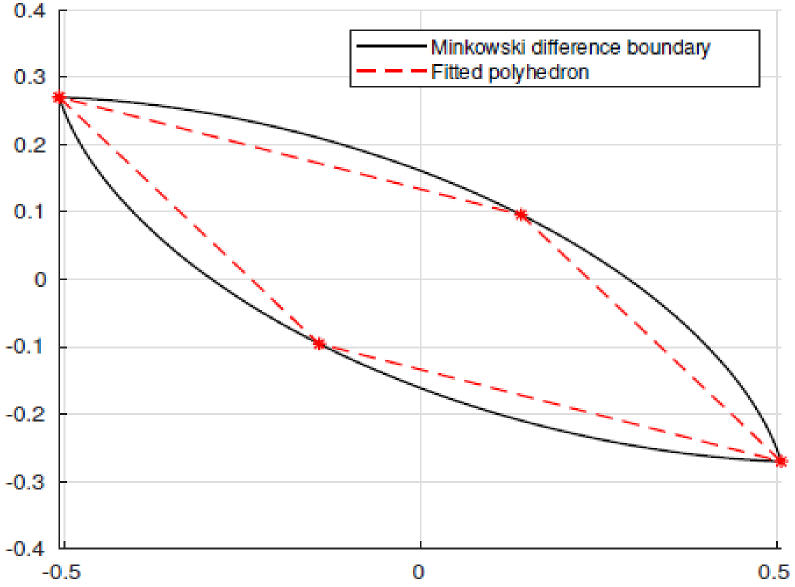
- Closed-form Minkowski difference between ellipsoids
  - Applied at each orientation of ellipsoid A
  - Shrink A to a sphere (Affine Transform) -> Offset curve in Shrunk Space -> Stretching back



Geometric Lower Bound based on the Closed-form Minkowski Difference

- Lower bounds of Minkowski difference boundary in Shrunk Space
  - Extreme distance at each axis of Ellipsoid that the Sphere can reach (N-dim)
  - Construct polyhedron at each orientation
  - Affine transform preserves convexity
- Geometric Lower Bound
  - union of the polyhedron at each orientation





Containment Checking Process

- Point-in-polyhedron checking:
  - Decompose into disjoint simplexes by triangulation
  - For each simplex, query the configuration as a convex combination of the vertices of the simplex
    - $\begin{pmatrix} P \\ 1 \end{pmatrix} = \begin{pmatrix} P_{S_0} & P_{S_1} & \dots & P_{S_n} \\ 1 & 1 & \dots & 1 \end{pmatrix} (\lambda_0 \; \lambda_1 \; \dots \lambda_n)^T, \lambda_i \in [0,1] \; (\forall i)$
      - P: the point to be queried
      - P_{S_i}: i-th vertex that defines the simplex
- For Convex Lower Bound: directly apply "point-in-polyhedron" test
- For Geometric Lower Bound: with the knowledge of orientation, apply the test at the specific orientation

Volume of the Lower Bounds

- Volume of polyhedron
  - Triangulation -> disjoint simplexes
  - Volume: $V_{poly}(P) = \sum_{i=1}^{m} V_{simplex}(P_{S_0}, P_{S_1}, \ldots, P_{S_n})$
    - P: the point to be queried
    - P_{S_i}: i-th vertex that defines the simplex
    - m: number of disjoint simplex
    - n: dimension of the space
    - $V_{simplex} = \left\| \frac{1}{n!} \det(P_{S_1} - P_{S_0}, P_{S_2} - P_{S_0}, \ldots, P_{S_n} - P_{S_0}) \right\|$
- For Convex Lower Bound: directly compute the volume of polyhedron
- For Geometric Lower Bound: integral of volume at each orientation
  - $V_{total} = \int_R V_{geo}(R) dR$
    - $V_{geo}(R)$: volume of the geometric lower bound at each orientation
    - dR: Haar measure for integration on $\mathrm{SO}(n)$

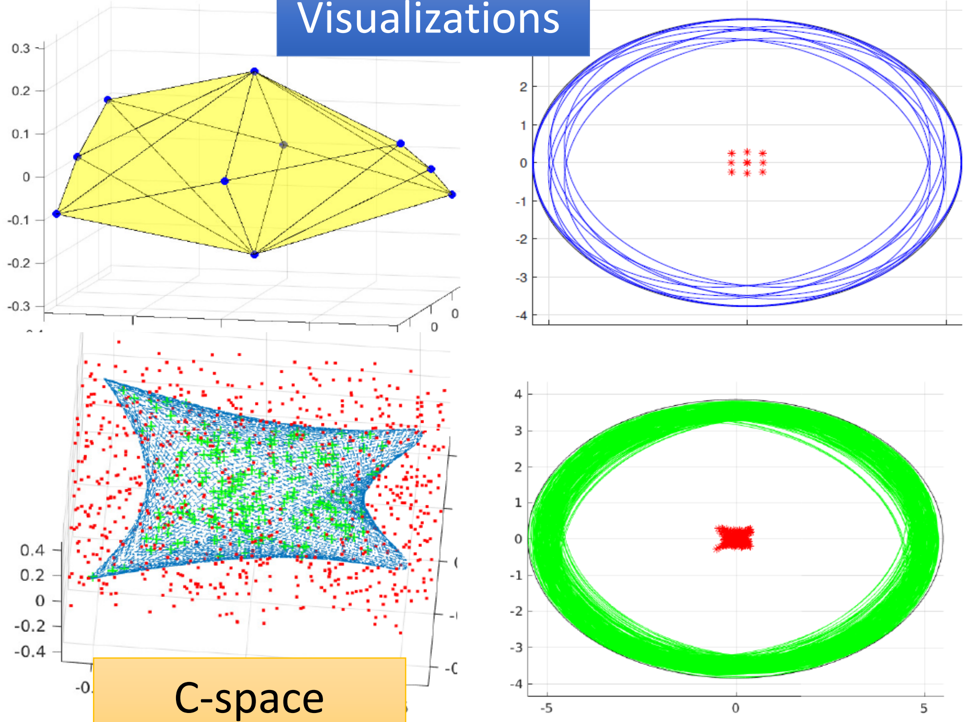
2D Validation

- Shape of the lower bounds
- Containment checking validation
- Volume comparisons
  - inflation factor ε
    - $\boldsymbol{b} = (1+\varepsilon)\boldsymbol{a}$
  - aspect ratio α
    - $\alpha = a_1/a_2$

Visualizations

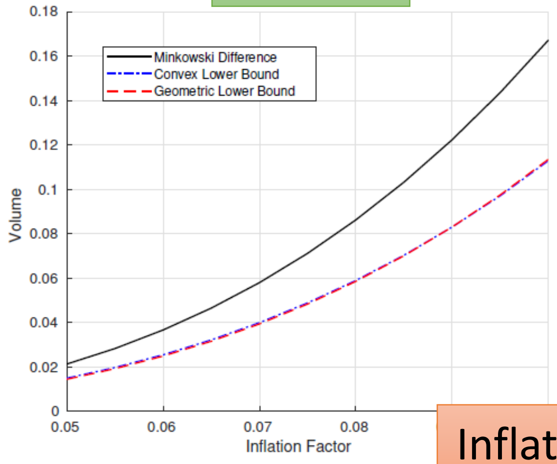
Convex Lower Bound

Geometric Lower Bound

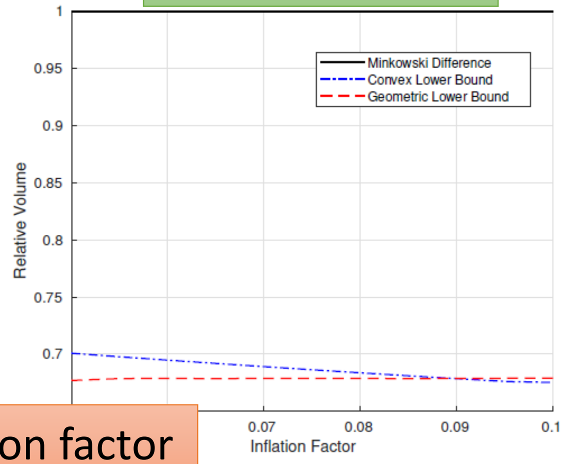


C-space

Volume

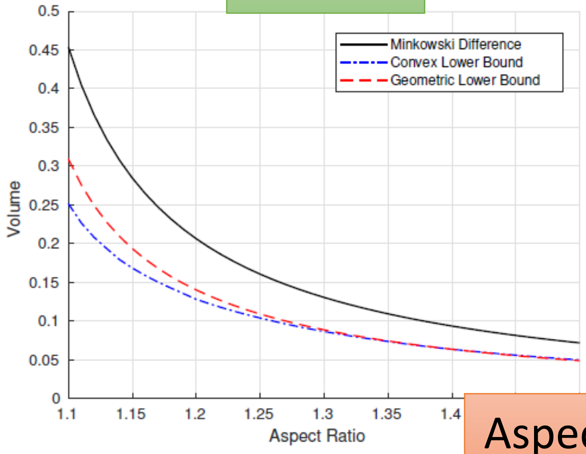


Relative Volume

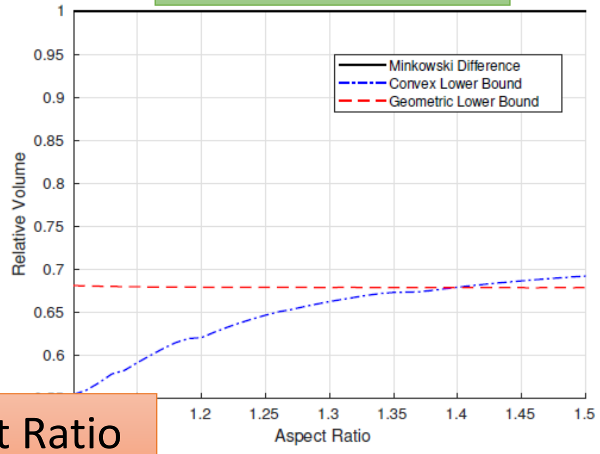


Inflation factor

Volume



Relative Volume



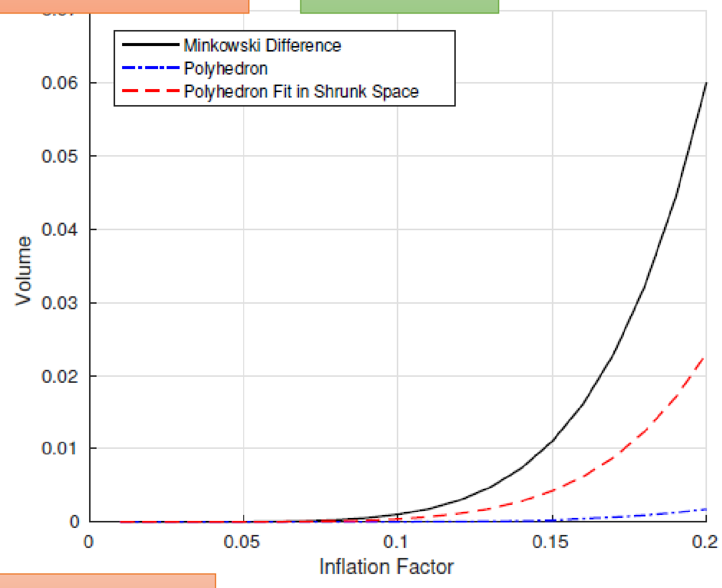
Aspect Ratio

3D Validation

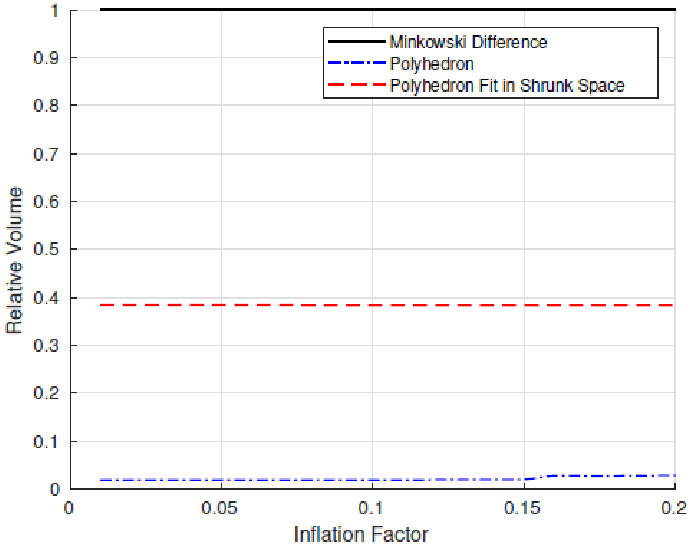
- Volume comparisons
  - Inflation factor ε
    - $\boldsymbol{b} = (1+\varepsilon)\boldsymbol{a}$
  - Aspect ratio α_1, α_2
    - $\alpha_1 = a_1/a_2$
    - $\alpha_2 = a_1/a_3$

Inflation factor

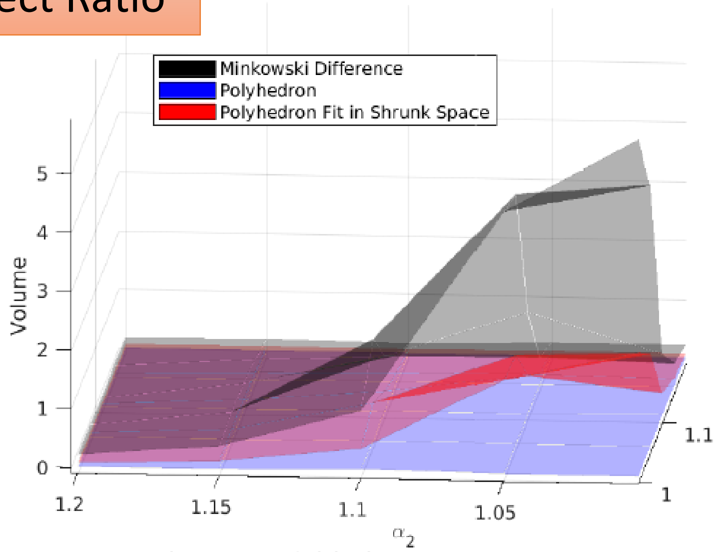
Volume

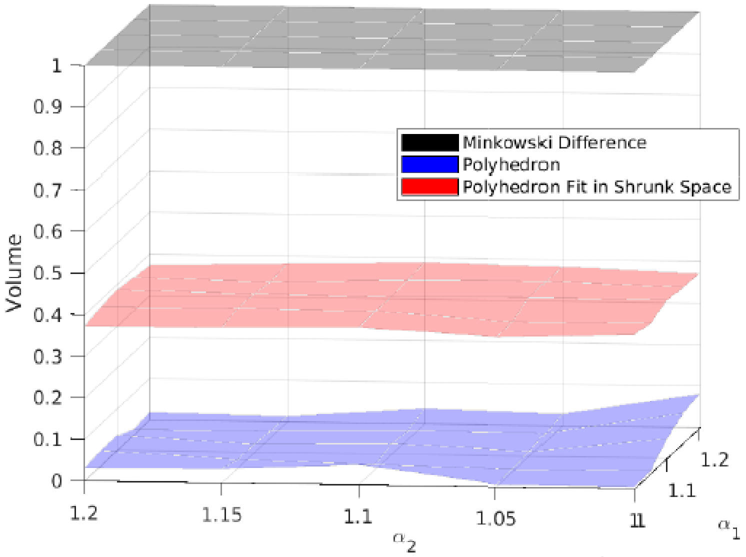


Relative Volume



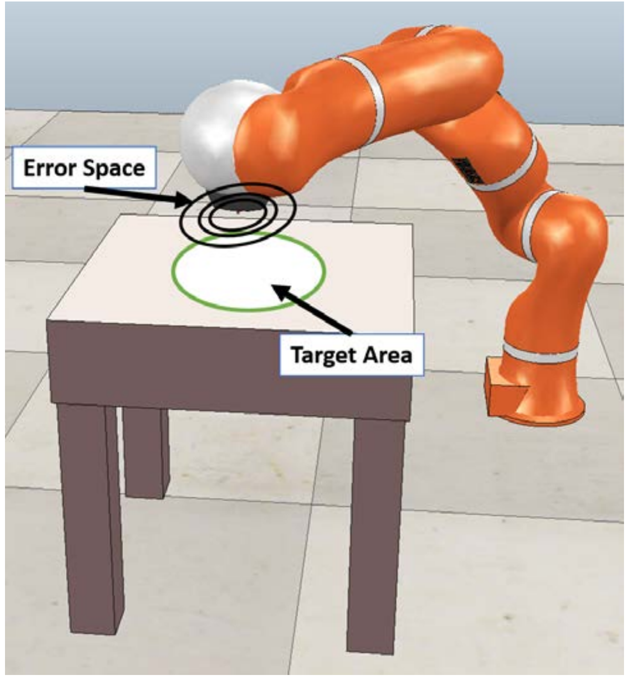
Aspect Ratio

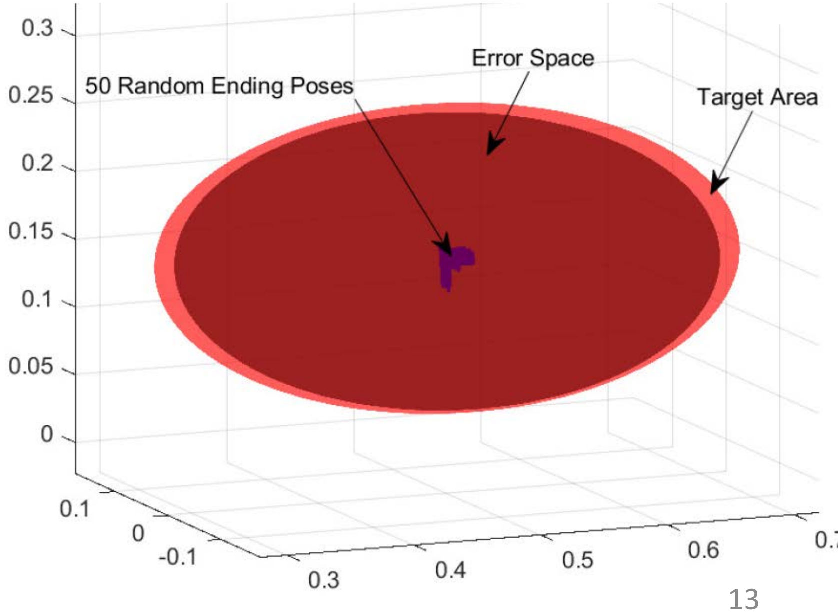




Potential Applications

- Assembly task (KUKA LWR robot)
  - Target pose: $(R|\mathbf{t}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0.15 \end{array}\right)$
  - Joint space:
    - $[-0.7768, 0.1991, -0.1991, 1.6981, -1.6241, 1.9656, -0.9147]^T$
  - Shape of an object: Ellipsoid with semi-axis length $\mathbf{a}_0 = [0.2, 0.15, 0.1]^T$
  - Decide how large space the end effector can move
    - inflated by $\varepsilon = 0.1$
    - Geometric Lower Bound Volume: $V_{total} = 5.2385 \times 10^{-8}$
    - error tolerance of each joint





Conclusions

- Two main goals: Containment checking and volume of the allowable motions in C-space
- **Convex Lower Bound**: a convex polyhedron subspace in Configuration space based on Algebraic Condition of Containment
- **Geometric Lower Bound**: a union of polyhedron subspace based on the Minkowski difference between two ellipsoids
- Validated the theory by 2D and 3D examples
- Discussed potential applications

Acknowledgements

- The authors thank the following people for useful discussions:
  - Dr. Jin Seob Kim, Dr. Arash Ghaani Farashahi, Dr. Shengnan Lu, and Ms. Mengdi Xu
- This work is supported by:
  - National Science Foundation grant IIS-1619050
  - Office of Naval Research Award N00014-17-1-2142

Thank You !