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Pose Changes from a Different Point of View

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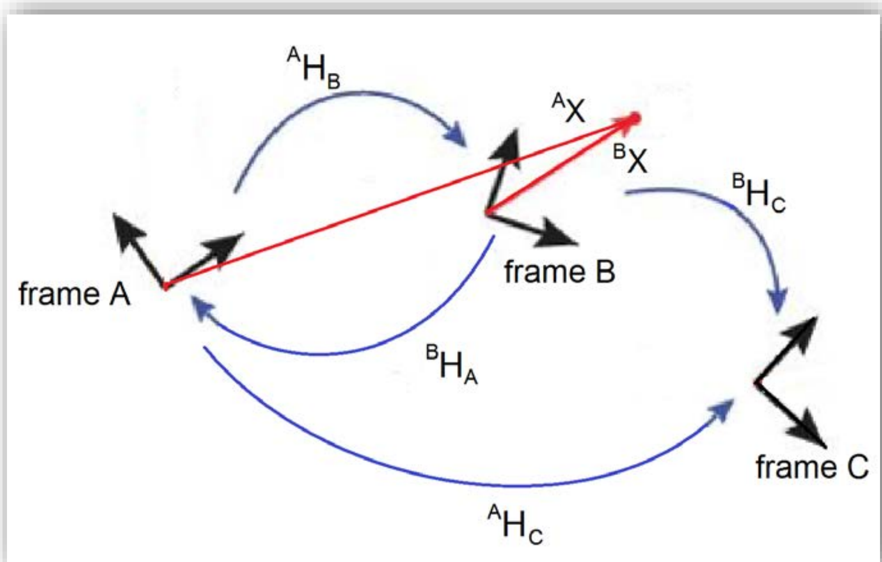
Content

- Traditional Point of View
- Concept Definitions
- Conjugations and Change of View
- Screw Parameters
- Introduction to “Pose Change Group” (PCG)
- Applications
- Conclusions

Traditional View

- Describe rigid body motions as homogeneous transformations
  - ${}^{A}H_{B}$: A --> B as seen in frame A
  - ${}^{A}H_{C} = {}^{A}H_{B}{}^{B}H_{C}$: Concatenation of rigid body motions
  - ${}^{B}H_{A} = ({}^{A}H_{B})^{-1}$: Inverse transformation
  - ${}^{A}x = {}^{A}H_{B}\,{}^{B}x$: Vector conversion between two coordinate systems



Traditional View

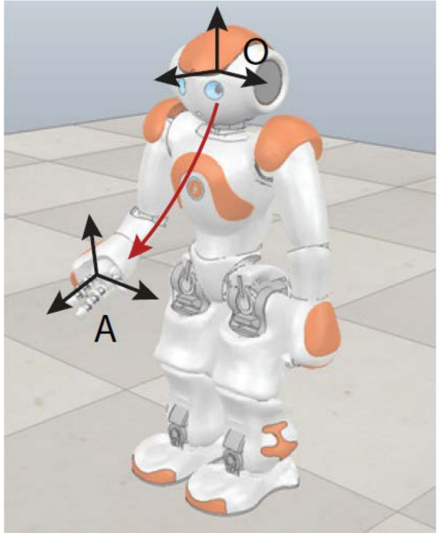
- The group of rigid-body motion, i.e. SE(3)
  - $(R, \mathbf{t})$: Rotation-translation pair
  - $(R_1, \mathbf{t}_1) \circ (R_2, \mathbf{t}_2) = (R_1 R_2, R_1 \mathbf{t}_2 + \mathbf{t}_1)$
    - "Semi-direct" product
  - $(R, \mathbf{t})^{-1} = (R^T, -R^T \mathbf{t})$
- Group action
  - $(R, \mathbf{t}) \cdot \boldsymbol{x} = R\boldsymbol{x} + \mathbf{t}$
  - $[(R_1, \mathbf{t}_1) \circ (R_2, \mathbf{t}_2)] \cdot \boldsymbol{x} = (R_1, \mathbf{t}_1) \cdot [(R_2, \mathbf{t}_2) \cdot \boldsymbol{x}]$

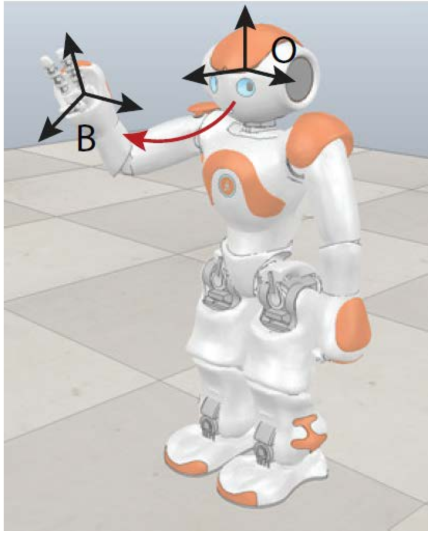
$R \in$ SO(3), $\boldsymbol{t} \in \mathrm{R}^3$
"∘" : Group operator
"T" : Matrix transpose
"-1" : Group inverse

$\mathbf{x} \in \mathrm{R}^3$, $(R, \boldsymbol{t}) \in$ SE(3)

Traditional View

- A natural way for motion descriptions, commonly used for a century in kinematics community
- Hard to model a motion between two poses as seen in a third point of view
  - a humanoid robot seeing his hand moving
  - a ground-based remote control of aerial vehicles



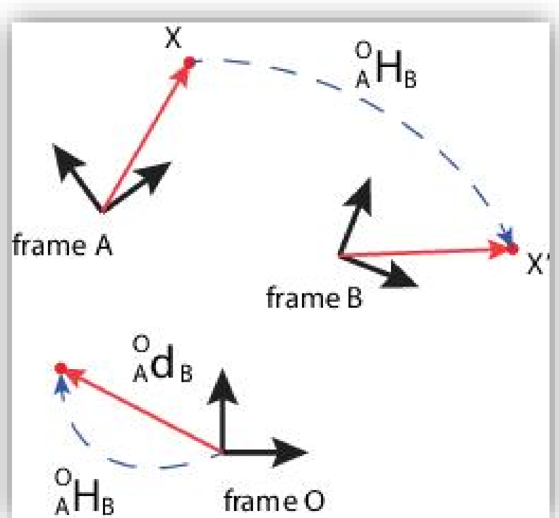


Concept Definitions

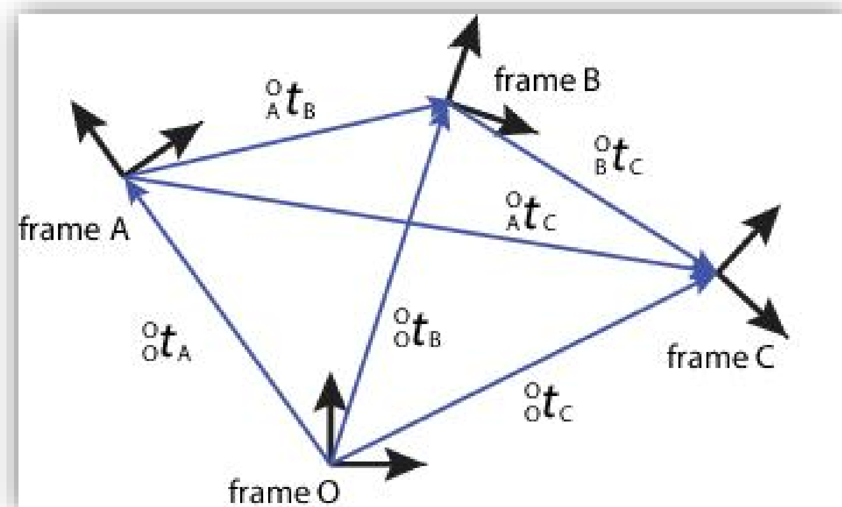
- Euclidean Motion
  - Handedness-preserving isometry of Euclidean space
  - Can be described using group-theoretic tools
  - Well-defined without coordinates or frames of reference
- Pose
  - Describes a position and orientation in space
- Change of Pose
  - Conversion of a pose into another
  - A movement of frame without moving the whole space
- Coordinate Transformation
  - Describe changes in the space-fixed frame
  - Describe changes in the way frames are attached to moving body

Conjugation and Change of View

- Problem of interest
  - ${}^{O}_{A}H_B = H({}^{O}_{A}R_B, {}^{O}_{A}\mathbf{d}_B)$
  - Note: ${}^{O}_{A}\mathbf{d}_B$ is **not** the translation vector ${}^{O}_{A}\mathbf{t}_B$, but the displacement of frame origin O under Euclidean motion ${}^{O}_{A}H_B$



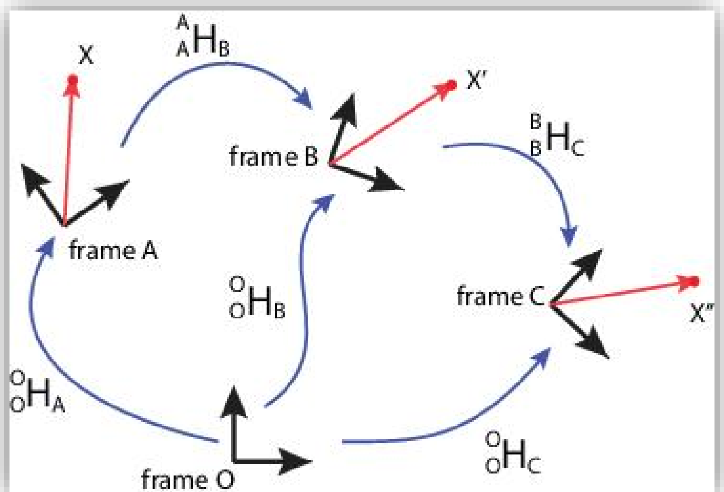
The meaning of ${}^{O}_{A}\mathbf{d}_B$



Translation vectors of different frame origins

Conjugation and Change of View

- Relationship between ${}^{O}_{A}H_{B}$, ${}^{A}_{A}H_{B}$, ${}^{O}_{O}H_{A}$
  - Let $\boldsymbol{x}$ be a point in space, which can be described in **any** reference frame
  - Coordinate transformation of a point: ${}^{O}X = {}^{O}\mathcal{H}_{A}\,{}^{A}X$
  - Euclidean motion of the point: ${}^{O}X' = {}^{O}_{A}H_{B}\,{}^{O}X$



Relationship between frames O, A, B and C

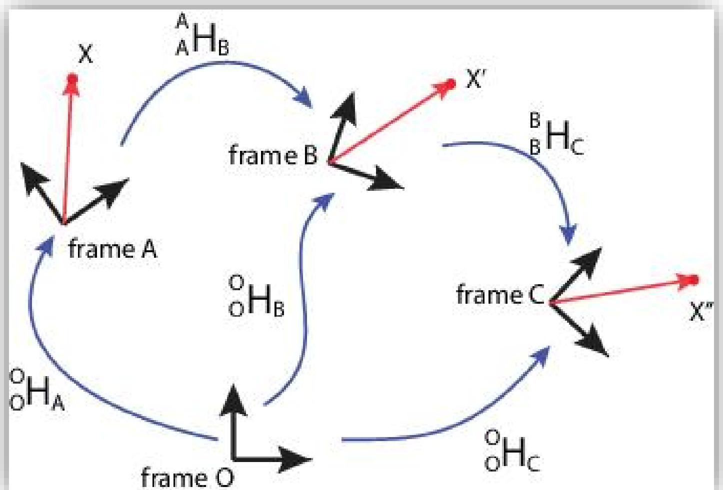
$$
\begin{aligned}
{}^{A}X' &= {}^{A}_{A}H_{B}\,{}^{A}X \\
{}^{A}X' &= {}^{A}\mathcal{H}_{B}\,{}^{B}X' \\
{}^{B}X' &= {}^{A}X \\
\Rightarrow {}^{A}\mathcal{H}_{B} &= {}^{A}_{A}H_{B}
\end{aligned}
$$

Conjugation and Change of View

- Conjugation: $^{O}_{A}H_{B} = {}^{O}\mathcal{H}_{A}\ {}^{A}_{A}H_{B}\ {}^{O}\mathcal{H}_{A}^{-1} = {}^{O}_{O}H_{A}\ {}^{A}_{A}H_{B}\ {}^{O}_{O}H_{A}^{-1}$
  - Rotation part: $^{O}_{A}R_{B} = {}^{O}_{O}R_{A}\ {}^{A}_{A}R_{B}\ {}^{O}_{O}R_{A}^{-1}$
  - Translation part: $\boxed{^{O}_{A}\mathbf{d}_{B}} = (I - {}^{O}_{A}R_{B})\ {}^{O}_{O}\mathbf{t}_{A} + \boxed{^{O}_{A}\mathbf{t}_{B}}$

$^{O}_{A}\mathbf{t}_{B} = {}^{O}_{O}R_{A}\ {}^{A}_{A}\mathbf{t}_{B}$

- Transformation of concatenated displacement
  - $^{O}_{A}H_{C} = {}^{O}_{B}H_{C}\ {}^{O}_{A}H_{B}$
    - In contrast with rigid-body kinematics: $^{A}_{A}H_{C} = {}^{A}_{A}H_{B}\ {}^{B}_{B}H_{C}$



Relationship between frames O, A, B and C

Screw Parameters

- Expression of any Euclidean motion:
  - $H(\mathbf{n}, \mathbf{p}, \theta, d) = \begin{pmatrix} e^{\theta N} & (I - e^{\theta N})\mathbf{p} + d\mathbf{n} \\ \mathbf{0}^T & 1 \end{pmatrix}$
    - $\mathbf{n}$: Direction of screw axis
    - N: Skew-symmetric matrix corresponding to $\mathbf{n}$, $N = \hat{\mathbf{n}}$
      - Can be obtained by: $R - R^T = 2 \sin\theta \, N$
    - $\mathbf{p}$: Unique vector pointing to the axis, s.t. $\mathbf{p} \cdot \mathbf{n} = 0$
    - θ: Angle of rotation
      - $\theta = \arccos(\frac{trace(R) - 1}{2})$
    - d: Distance along the axis
      - $d = \mathbf{d} \cdot \mathbf{n}$
      - ${}_A d_B = {}^O_A\mathbf{d}_B \cdot {}^O_A\mathbf{n}_B = {}^O_A\mathbf{t}_B \cdot {}^O_A\mathbf{n}_B$
      - Can be obtained by ${}^O_A H_B = H({}^O_A R_B, {}^O_A\mathbf{d}_B)$ or $({}^O_A R_B, {}^O_A\mathbf{t}_B)$

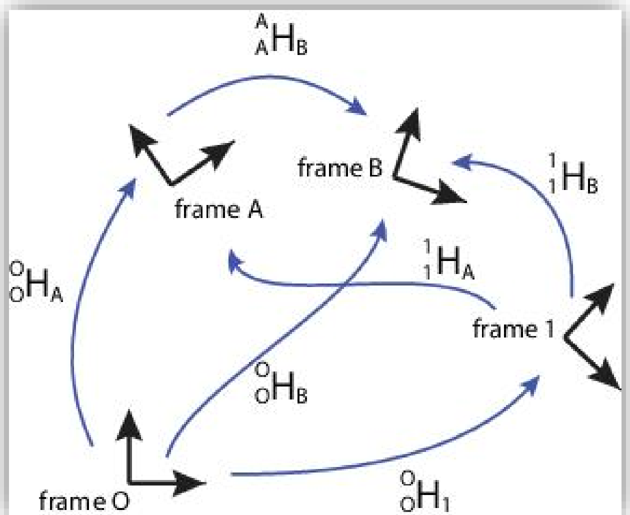
(${}_A\theta_B$, ${}_A d_B$) are invariant to the choice of O

Pose Change Group

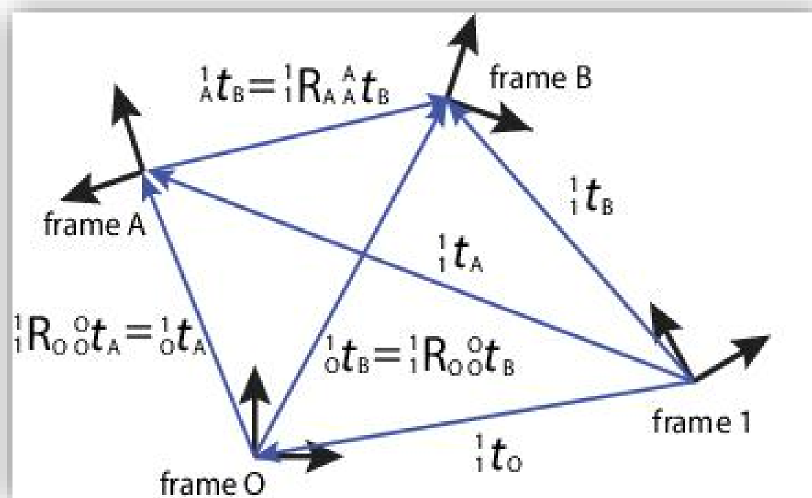
- Definition of "direct-product" group: $\mathrm{PCG}(3) = \mathrm{SO}(3) \times \mathbb{R}^3$
  - $({}^{O}_{A}R_C, {}^{O}_{A}\mathbf{t}_C) = ({}^{O}_{B}R_C, {}^{O}_{B}\mathbf{t}_C) \cdot ({}^{O}_{A}R_B, {}^{O}_{A}\mathbf{t}_B) = ({}^{O}_{B}R_C\,{}^{O}_{A}R_B, {}^{O}_{B}\mathbf{t}_C + {}^{O}_{A}\mathbf{t}_B)$
    - ${}^{O}_{A}R_B = \exp\left({}_{A}\theta_B\ {}^{O}_{A}\hat{\mathbf{n}}_B \right) = {}^{O}_{O}R_A\,{}^{A}_{A}R_B\,{}^{O}_{O}R_A^T$
    - ${}^{O}_{A}\mathbf{t}_B$ is the displacement from A to B as seen in O
- Let $\boldsymbol{x}$ denote a point in space
  - ${}^{O}_{A}\mathbf{t}_X$: position of $\boldsymbol{x}$ from the origin of A as seen in O
- Group action on pose space: $(Q, \xi) \odot (R, \mathbf{t}) \doteq \left(QRQ^T, Q\mathbf{t}\right)$

Pose Change Group

- Change of observer frame
  - If the motion is viewed from frame 1 rather than frame O:
    - Transformation: ${}^{1}_{A}H_{B} = {}^{O}_{O}H_{1}^{-1}\,{}^{O}_{A}H_{B}\,{}^{O}_{O}H_{1} = {}^{1}_{1}H_{O}\ {}^{O}_{A}H_{B}\,{}^{1}_{1}H_{O}^{-1}$
    - Rotational and translational parts can be obtained by group action
      - $\left({}^{1}_{A}R_{B}, {}^{1}_{A}\mathbf{t}_{B}\right) = \left({}^{1}_{1}R_{O}, \mathbf{0}\right) \odot \left({}^{O}_{A}R_{B}, {}^{O}_{A}\mathbf{t}_{B}\right)$
      - ${}^{1}_{A}R_{B} = {}^{1}_{1}R_{O}\ {}^{O}_{A}R_{B}\ {}^{1}_{1}R_{O}^{T}$
      - ${}^{1}_{A}\mathbf{t}_{B} = {}^{1}_{1}R_{O}\ {}^{O}_{A}\mathbf{t}_{B}$



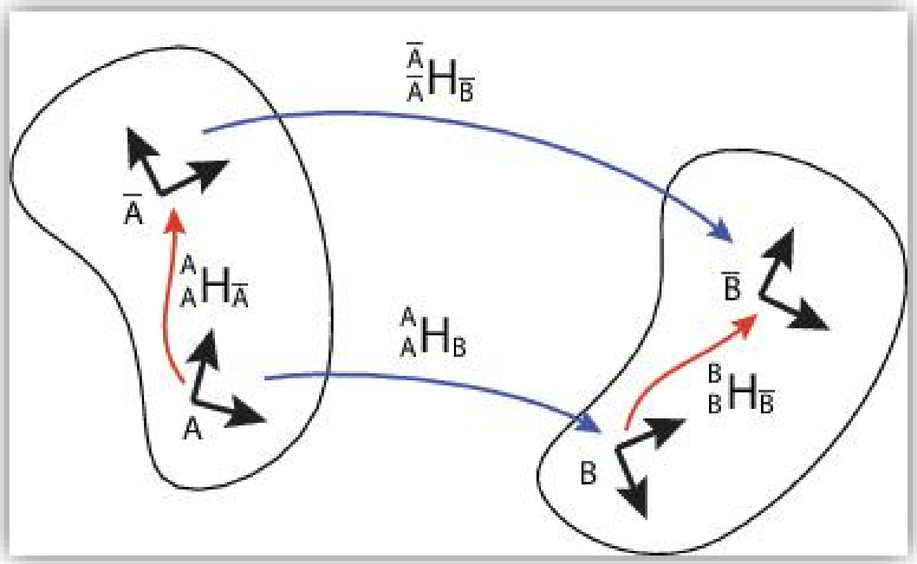
Transformation of the observer frame from O to 1



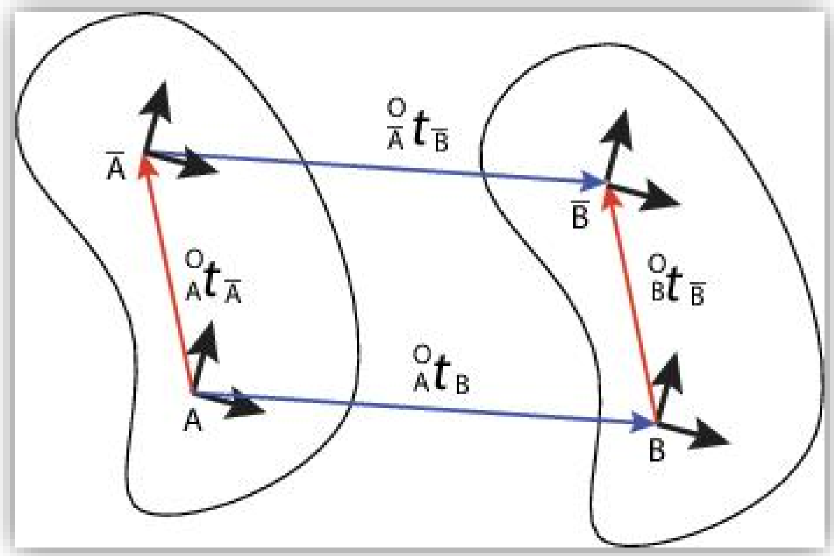
Coordinates of all vectors calculated in frame 1

Pose Change Group

- Change of body-fixed frame
  - $({}^{O}_{\bar{A}}R_{\bar{B}}, {}^{O}_{\bar{A}}\mathbf{t}_{\bar{B}}) = (R_0, {}^{O}_{B}\mathbf{t}_{\bar{B}}) \cdot ({}^{O}_{A}R_{B}, {}^{O}_{A}\mathbf{t}_{B}) \cdot (R_0, {}^{O}_{A}\mathbf{t}_{\boldsymbol{A}})^{-1}$
    - R_0 is defined by ${}^{O}_{O}H_{\bar{A}}\Delta^{-1} = H(R_0, \mathbf{d}_0)$
    - $\Delta \doteq {}^{A}_{A}H_{\bar{A}} = {}^{B}_{B}H_{\bar{B}}$



Conjugation resulting from changing the body-fixed frame



Translational part of the pose changes

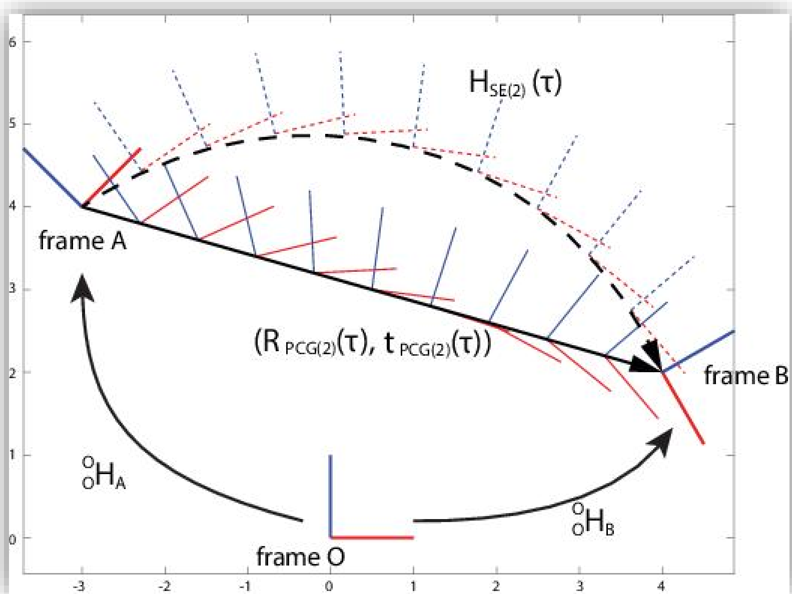
Applications

- Bi-invariant Metrics
  - Definition: $d(g_1, g_2) = d(h \circ g_1, h \circ g_2) = d(g_1 \circ k, g_2 \circ k)$
  - SE(3) does not have nontrivial bi-invariant metric functions
    - $d_{SE(3)}\left(H_0 H_1 H_0^{-1}, H_0 H_2 H_0^{-1}\right) \neq d_{SE(3)}(H_1, H_2) \neq d_{SE(3)}(H_1 H_0, H_2 H_0)$
  - PCG(3) tolerates bi-invariance under direct-product
    - $d_{PCG(3)}\big((R_0, \mathbf{t}_0) \cdot (R_1, \mathbf{t}_1), (R_0, \mathbf{t}_0) \cdot (R_2, \mathbf{t}_2)\big) = d_{PCG(3)}\big((R_1, \mathbf{t}_1), (R_2, \mathbf{t}_2)\big) = d_{PCG(3)}\big((R_1, \mathbf{t}_1) \cdot (R_0, \mathbf{t}_0), (R_2, \mathbf{t}_2) \cdot (R_0, \mathbf{t}_0)\big)$
  - Plays a role in interpolating paths between two poses or motions

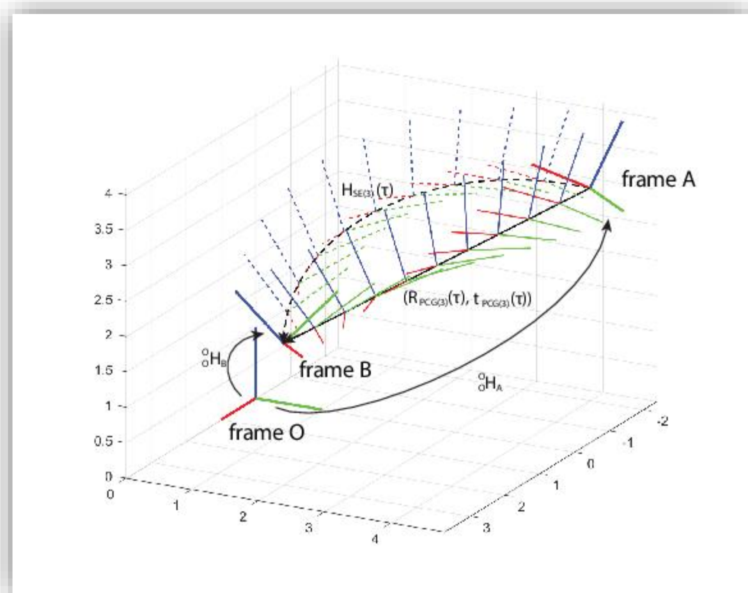
Applications

- Path Generations
  - SE(3): $H(\tau) = \exp(\tau \log({}^{O}_{A}H_B))\,{}^{O}_{O}H_A$
  - PCG(3): $(R(\tau), \mathbf{t}(\tau)) = (\exp\left(\tau \log({}^{O}_{A}R_B)\right), {}^{O}_{A}\mathbf{t}_B \tau) \cdot ({}^{O}_{O}R_A, {}^{O}_{O}\mathbf{t}_A)$

Trajectory starts at ${}^{O}_{O}H_A$ at $\tau = 0$, and ends at ${}^{O}_{O}H_B$ at $\tau = 1$



2D Case



3D Case

Conclusions

- Differences between Pose Changes and Euclidean Motions are reviewed
- An analytical framework for composition of pose changes as a direct product operation is developed
- Applications in bi-invariant metric and path generation are introduced

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Thank You !