Pose Changes from a Different Point of View

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Traditional View

- Describe rigid body motions as homogeneous transformations
  - $^A H_B : A \rightarrow B$ as seen in frame A
  - $^A H_C = ^A H_B \cdot ^B H_C$ : Concatenation of rigid body motions
  - $^B H_A = (^A H_B)^{-1}$ : Inverse transformation
  - $^A x = ^A H_B \cdot ^B x$ : Vector conversion between two coordinate systems
Traditional View

• The group of rigid-body motion, i.e. SE(3)
  – \((R, t)\) : Rotation-translation pair
  – \((R_1, t_1) \circ (R_2, t_2) = (R_1 R_2, R_1 t_2 + t_1)\)
    • “Semi-direct” product
  – \((R, t)^{-1} = (R^T, -R^T t)\)

• Group action
  – \((R, t) \cdot x = Rx + t\)
  – \([ (R_1, t_1) \circ (R_2, t_2) ] \cdot x = (R_1, t_1) \cdot [ (R_2, t_2) \cdot x ]\)
Traditional View

• A natural way for motion descriptions, commonly used for a century in kinematics community

• Hard to model a motion between two poses as seen in a third point of view
  – a humanoid robot seeing his hand moving
  – a ground-based remote control of aerial vehicles
Concept Definitions

• Euclidean Motion
  – Handness-preserving isometry of Euclidean space
  – Can be described using group-theoretic tools
  – Well-defined without coordinates or frames of reference

• Pose
  – Describes a position and orientation in space

• Change of Pose
  – Conversion of a pose into another
  – A movement of frame without moving the whole space

• Coordinate Transformation
  – Describe changes in the space-fixed frame
  – Describe changes in the way frames are attached to moving body
Conjugation and Change of View

• Problem of interest
  
  - \( \overset{O}{H}_B = H(\overset{O}{R}_B, \overset{O}{d}_B) \)
  
  - Note: \( \overset{O}{d}_B \) is **not** the translation vector \( \overset{O}{t}_B \), but the displacement of frame origin O under Euclidean motion \( \overset{O}{H}_B \)

The meaning of \( \overset{O}{d}_B \)

Translation vectors of different frame origins
Conjugation and Change of View

- Relationship between $O_H^A B$, $A_H^A B$, $O_H^O A$
  - Let $x$ be a point in space, which can be described in any reference frame
  - Coordinate transformation of a point: $O_X = O_H^O A X$
  - Euclidean motion of the point: $O_X' = A_H^A O_X$

Relationship between frames O, A, B and C
Conjugation and Change of View

• Conjugation:

\[ \mathcal{O}_A H_B = \mathcal{O}_A H_A \mathcal{A}_A H_B \mathcal{O}_A H_A^{-1} = \mathcal{O}_O H_A \mathcal{A}_A H_B \mathcal{O}_O H_A^{-1} \]

  – Rotation part:

\[ \mathcal{O}_A R_B = \mathcal{O}_O R_A \mathcal{A}_A R_B \mathcal{O}_O R_A^{-1} \]

  – Translation part:

\[ \mathcal{O}_A d_B = (I - \mathcal{O}_A R_B) \mathcal{O}_O t_A + \mathcal{O}_A t_B \]

\[ \mathcal{O}_A t_B = \mathcal{O}_O R_A \mathcal{A}_A t_B \]

• Transformation of concatenated displacement

\[ \mathcal{O}_A H_C = \mathcal{O}_B H_C \mathcal{A}_A H_B \]

  • In contrast with rigid-body kinematics:

\[ \mathcal{A}_A H_C = \mathcal{A}_A H_B \mathcal{B} H_C \]

Relationship between frames O, A, B and C
Screw Parameters

• Expression of any Euclidean motion:

\[ H(n, p, \theta, d) = \begin{pmatrix} e^{\theta N} & (I - e^{\theta N})p + d n \\ 0^T \\ 1 \end{pmatrix} \]

- \( n \): Direction of screw axis
- \( N \): Skew-symmetric matrix corresponding to \( n \), \( N = \hat{n} \)
  - Can be obtained by: \( R - R^T = 2 \sin \theta \ N \)
- \( p \): Unique vector pointing to the axis, s.t. \( p \cdot n = 0 \)
- \( \theta \): Angle of rotation
  - \( \theta = \arccos\left(\frac{\text{trace}(R) - 1}{2}\right) \)
- \( d \): Distance along the axis
  - \( d = d \cdot n \)
  - \( A d_B = A^0 d_B \cdot A^0 n_B = A^0 t_B \cdot A^0 n_B \)
  - Can be obtained by \( A^0 H_B = H(A^0 R_B , A^0 d_B) \) or \( (A^0 R_B , A^0 t_B) \)

( \( A \theta_B, A d_B \) ) are invariant to the choice of \( O \)
Pose Change Group

• Definition of “direct-product” group: \( \text{PCG}(3) = \text{SO}(3) \times \mathbb{R}^3 \)
  \[ (O^\text{AR}_C, O^\text{At}_C) = (O^\text{BR}_C, O^\text{At}_C) \cdot (O^\text{AR}_B, O^\text{At}_B) = (O^\text{BR}_C O^\text{AR}_B, O^\text{At}_B + O^\text{At}_B) \]
  • \( O^\text{AR}_B = \exp(A^\text{AB} B^\text{AB} n_B) = O^\text{AR}_A O^\text{AR}_B O^\text{AR}_B^T \)
  • \( O^\text{At}_B \) is the displacement from A to B as seen in O

• Let \( x \) denote a point in space
  \[ O^\text{At}_A : \text{position of } x \text{ from the origin of } A \text{ as seen in } O \]

• Group action on pose space: \( (Q, \xi) \odot (R, t) = (QRQ^T, Qt) \)
Pose Change Group

• Change of observer frame
  – If the motion is viewed from frame 1 rather than frame O:
    • Transformation: $\mathbf{A}_1 H_B = \mathbf{O}_H^{-1} \mathbf{A}_B \mathbf{O}_H \mathbf{H}_1 = \mathbf{H}_O \mathbf{A}_B \mathbf{H}_O^{-1}$
    • Rotational and translational parts can be obtained by group action
      \begin{align*}
      \begin{pmatrix}
      1 & R_B \\
      1 & t_B
      \end{pmatrix}
      &= \begin{pmatrix}
      1 & R_O \\
      1 & t_O
      \end{pmatrix} \circ \begin{pmatrix}
      1 & R_B \\
      1 & t_B
      \end{pmatrix} \\
      1 & R_B &= 1 & R_O \mathbf{A}_B \mathbf{R}_O \\
      1 & t_B &= 1 & R_O \mathbf{A}_t_B
      \end{align*}

Transformation of the observer frame from O to 1

Coordinates of all vectors calculated in frame 1
Pose Change Group

• Change of body-fixed frame

\[- \left( ^O_A \mathbf{R}_B, ^O_A \mathbf{t}_B \right) = \left( R_0, ^O_B \mathbf{t}_B \right) \cdot \left( ^O_A \mathbf{R}_B, ^O_A \mathbf{t}_B \right) \cdot \left( R_0, ^O_A \mathbf{t}_A \right)^{-1} \]

- $R_0$ is defined by $^O_0 \mathbf{H}_A \Delta^{-1} = \mathbf{H}(R_0, \mathbf{d}_0)$
- $\Delta \doteq \frac{A}{A} \mathbf{H}_A = \frac{B}{B} \mathbf{H}_B$

Conjugation resulting from changing the body-fixed frame

Translational part of the pose changes
Applications

• Bi-invariant Metrics
  – Definition: \( d(g_1, g_2) = d(h \circ g_1, h \circ g_2) = d(g_1 \circ k, g_2 \circ k) \)
  – SE(3) does not have nontrivial bi-invariant metric functions
    • \( d_{SE(3)}(H_0 H_1 H_0^{-1}, H_0 H_2 H_0^{-1}) \neq d_{SE(3)}(H_1, H_2) \neq d_{SE(3)}(H_1 H_0, H_2 H_0) \)
  – PCG(3) tolerates bi-invariance under direct-product
    • \( d_{PCG(3)}((R_0, t_0) \cdot (R_1, t_1), (R_0, t_0) \cdot (R_2, t_2)) = d_{PCG(3)}((R_1, t_1), (R_2, t_2)) = d_{PCG(3)}((R_1, t_1) \cdot (R_0, t_0), (R_2, t_2) \cdot (R_0, t_0)) \)
  – Plays a role in interpolating paths between two poses or motions
Applications

• Path Generations
  - SE(3): \( H(\tau) = \exp(\tau \log(OAHB))OHA \)
  - PCG(3): \( (R(\tau), t(\tau)) = \left( \exp\left(\tau \log(OAR_B)\right), OA t_B \tau \right) \cdot \left(OAR_A, OT_A\right) \)

Trajectory starts at \( OHA \) at \( \tau = 0 \), and ends at \( OHB \) at \( \tau = 1 \)
Conclusions

• Differences between Pose Changes and Euclidean Motions are reviewed
• An analytical framework for composition of pose changes as a direct product operation is developed
• Applications in bi-invariant metric and path generation are introduced
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